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RESEARCH REPORT No. EM-41

DIFFRACTION OF ELECTROMAGNETIC WAVES BY A PLANE WIRE GRATING II

by

WILLIAM SOLLFREY

and

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Abstract

Variational expressions for the scattering matrix of a wire grating obtained in a previous report (EM-18) are evaluated for the case of circularly cylindrical wires. The single scattering approximation is used as a trial field.

1. Introduction

In a previous report[1], stationary expressions for the scattering matrix elements of a wire grating were derived. These expressions do not involve any assumptions about the size or shape of the wires. In that report, these stationary expressions were evaluated for the case of thin wires, in which the shape does not play any role. In the present report, which should be considered as an appendix to the previous one*, these expressions will be evaluated for the specific shape of the circular cylinder, but with no restrictions on size. The trial current distributions will be taken from the solution of the problem of diffraction of a plane wave by a single circular cylindrical wire. The paper of Schaefer and Reiche[2] derives the intensities of the various orders from that solution directly. In this paper the results are improved by using the current distribution obtained by neglecting interaction in a variational expression.

The Green's function to be used in this analysis is identical with that derived previously, but it is expressed differently. Thus, instead of looking for a solution of

$$(\nabla^2 + k^2)G(s;x,z;x',z') = \sum_{-\infty}^{\infty} \exp(inska\sin\theta)\delta(x-x'-na)\delta(z-z') \qquad (3)$$

with the appropriate periodicity, we recognize that the solution of the wave equation with the specified set of inhomogeneous terms is the sum of the solutions of the wave equation with only one inhomogeneous term. Thus we have

$$G(s;r;r') = \sum_{-\infty}^{\infty} G^n\,(s;r;r') \qquad (74a)$$

$$(\nabla^2+k^2)G^n = \exp(inska\sin\theta_1)\delta(x-x'-na)\delta(z-z') \qquad (74b)$$

This equation is readily solved in polar coordinates,

$$G^n = \frac{i}{4}\,H_0^{(1)}\,(k|\vec{r} - \vec{na} - \vec{r}'|)\cdot e^{inska\sin\theta_1} \quad . \qquad (75)$$

2. TE Wave Problem

We shall now take up the TE wave problem. First let us calculate the current distribution excited in a single cylinder of radius R by a plane wave. In polar coordinates, with $\theta = 0$ the direction of the incident wave, we have

---

* The equations in this report are numbered consecutively with those in EM-18. References to equations (1) through (73) will be found in EM-18.

$$\begin{aligned}\phi &= \phi_{inc} + \phi_{sc} \\ &= \exp(ikr\cos\theta) + \sum_{-\infty}^{\infty} A_n H_n^{(1)}(kr)e^{in\theta}\end{aligned} \tag{76}$$

where the scattered field has been expanded in a series of eigenfunctions, the coefficients A_n to be determined. Expanding the incident field gives

$$\phi = \sum_{-\infty}^{\infty} \left[i^n J_n(kr) + A_n H_n^{(1)}(kr) \right] e^{in\theta} \quad . \tag{77}$$

Since the boundary condition is that ϕ vanish on the cylinder, $r = R$, we obtain

$$0 = i^n J_n(kR) + A_n H_n^{(1)}(kR) \quad . \tag{78}$$

Putting the values of the coefficients A_n thus obtained into (77), differentiating with respect to r, and finally evaluating this derivative at $r = R$,

$$\phi_r(R,\theta) = \sum_{-\infty}^{\infty} \frac{i^n}{H_n^{(1)}(kR)} W\left\{H_n^{(1)}(kR), J_n(kR)\right\} e^{in\theta} , \tag{79}$$

where W is the Wronskian of the two functions. Since the Wronskian of these two Bessel functions is $2/\pi Ri$, we take as the current distribution

$$I(\theta) = \sum_{-\infty}^{\infty} \frac{2i^{n-1}}{\pi R H_n^{(1)}(kR)} e^{in\theta} \quad . \tag{80}$$

Consider now the grating problem. Let the plane wave be incident at an angle θ_1; the proper current distribution is then $I(\theta-\theta_1)$. In the conjugate problem,

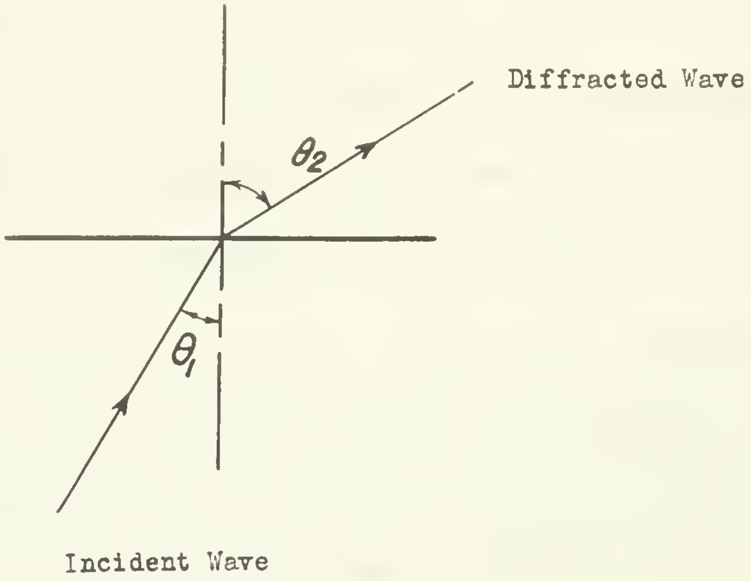

Figure 1.

the wave is incident in the direction $\pi + \theta_2$, so that the distribution is $I(\theta-\theta_2-\pi)$. The incident fields in the two cases are $\exp[ikr\cos(\theta-\theta_1)]$ and $\exp[-ikr\cos(\theta-\theta_2)]$ respectively.

We can now proceed to the evaluation of integrals in Eq. (19) The integrals in the numerator are

$$f_1 = \int_0^{2\pi} \exp[ikR\cos(\theta-\theta_2-\pi)]\, I(\theta-\theta_1)\, Rd\theta = f_1(\theta_1, \theta_2 + \pi) \tag{81a}$$

$$f_2 = \int_0^{2\pi} \exp[ikR\cos(\theta-\theta_1)]\, I(\theta-\theta_2-\pi)\, Rd\theta = f_1(\theta_2 + \pi, \theta_1) \tag{81b}$$

But, by a simple change of variables, (81a) becomes

$$f_1 = \int_0^{2\pi} \exp(ikR\cos\theta)\, I(\theta + \theta_2 + \pi - \theta_1)\, Rd\theta = f(\theta_2 - \theta_1) \tag{82}$$

a function of one variable only. Substituting (80) in (82), and interchanging the order of integration and summation, we get

$$f(\theta_2-\theta_1) = \sum_{-\infty}^{\infty} \frac{2i^{n-1}}{\pi H_n^{(1)}(kR)}\, e^{in(\pi+\theta_2-\theta_1)} \int_0^{2\pi} \exp(ikR\cos\theta) e^{in\theta} d\theta =$$

$$= \frac{4}{i} \sum_{-\infty}^{\infty} \frac{i^n e^{in(\theta_2-\theta_1+\pi)}}{H_n^{(1)}(kR)} \cdot i^{-n} J_{-n}(kR)$$

$$f(\theta_2-\theta_1) = -4i \sum_{-\infty}^{\infty} \frac{J_n(kR)}{H_n(kR)}\, e^{in(\theta_2-\theta_1)} \tag{83}$$

It is readily seen that the n-th term of $f_1(\theta_2-\theta_1)$ is identical with the (-n)-th term of $f_2(\theta_1-\theta_2)$, so that the two integrals are identical.

The next step is to evaluate the Green's function integral. Instead of retaining the form given by (75) and integrating around the same wire twice, we replace $\vec{r'} + \vec{na}$ by $\vec{\rho}$ thus shifting the integration with respect to ρ to the nth cylinder. The argument of the Hankel function in (75) is simply the wave number, k, times the distance between a point on the central cylinder and another point on the n-th cylinder.

In the 0-th term of the Green's function integral we still perform the integration around the same cylinder twice. Using the addition theorem for Bessel functions[3], we have

$$H_0^{(1)}(k|\vec{r}-\vec{\rho}|) = \sum_{m=-\infty}^{\infty} H_m^{(1)}(kR) J_m(kR) e^{im(\theta-\theta')} \tag{84}$$

where θ and θ' denote the angular coordinates of $\vec{r}$ and $\vec{\rho}$, respectively. We have then,

$$
\begin{aligned}
g_o &= \frac{i}{4}\iint I(\theta-\theta_2-\pi)H_o^{(1)}(k|\vec{r}-\vec{\rho}|)I(\theta'-\theta_1)R^2 d\theta d\theta' \\
&= \frac{i}{4}\sum_m H_m^{(1)}(kR)J_m(kR)\int_o^{2\pi} I(\theta-\theta_2-\pi)e^{im\theta}Rd\theta \int_o^{2\pi} I(\theta'-\theta_1)e^{-im\theta'}Rd\theta' \\
&= \frac{i}{4}\sum_m H_m^{(1)}(kR)J_m(kR)e^{im(\theta_2+\pi-\theta_1)}\int_o^{2\pi} I(\theta)e^{im\theta}Rd\theta \int_o^{2\pi} I(\theta)e^{-im\theta}d\theta \\
&= \frac{i}{4}\sum_m H_m^{(1)}(kR)J_m(kR)e^{im(\theta_2+\pi-\theta_1)}\,\frac{4i^{m-1}}{H_{-m}(kR)}\cdot\frac{4i^{-m-1}}{H_m(kR)} \\
&= -4i\sum_m \frac{J_m(kR)}{H_m(kR)}\,e^{im(\theta_2-\theta_1)} \qquad , \qquad (85)
\end{aligned}
$$

identical with the integral f. Thus we see that if we neglected the rest of the terms in the Green's function integral, the scattered wave amplitude would be essentially f. We should note that this is nothing but the far field amplitude scattered by a single cylinder from the same plane wave in the θ_2 direction. Neglecting the other terms is equivalent to assuming no interaction between grating cylinders. Since our trial current distribution was taken as the exact solution of the "no interaction" problem, the variational expression with a single term in the Green's function retained is the exact scattering matrix element for that problem. The rest of the terms will yield an estimate of the effect of interaction. We shall now evaluate these correction terms. In order to do this we need an appropriate expansion of the Hankel function. Using the addition formula for Bessel functions of order zero (for definition of angles and distances see Fig. 2)

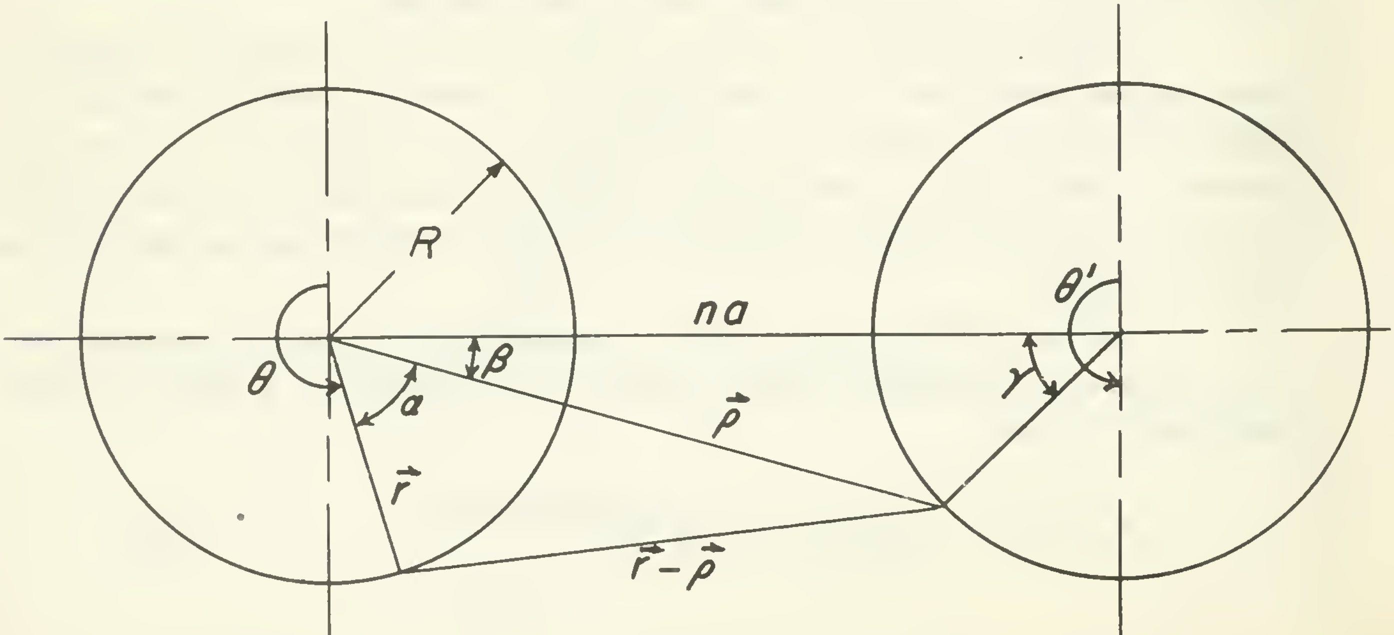

Figure 2. Two wires of the grating.

problem. The rest of the terms will yield an estimate of the effect of interaction. We shall now evaluate these correction terms. In order to do this we need an appropriate expansion of the Hankel function. Using the addition formula for Bessel functions of order zero (for definition of angles and distances see Fig. 2)

$$H_0^{(1)}(k|\vec{r}-\vec{\rho}|) = \sum_{\mu} H_{\mu}^{(1)}(k\rho)J_{\mu}(kR)e^{-i\mu\alpha}$$
$$= \sum_{\mu} J_{\mu}(kR)e^{i\mu(\frac{3\pi}{2}-\theta)} \cdot e^{i\mu\beta}H_{\mu}^{(1)}(k\rho)$$

Applying the addition theorem[3] to H_{μ},

$$e^{i\mu\beta}H_{\mu}^{(1)}(k\rho) = \sum_{\nu} H_{\mu+\nu}(kna)J_{\nu}(kR)e^{i\nu\gamma}$$

Hence

$$H_0^{(1)}(k|\vec{r}-\vec{\rho}|) = \sum_{\mu,\nu} i^{(\mu+\nu)}J_{\mu}(kR)J_{\nu}(kR)H_{\mu+\nu}(kna)e^{-i\mu\theta+i\nu\theta'} , \qquad (86)$$

and the n-th term of the Green's function integral becomes

$$e^{-inskasin\theta_1}g_{n+} = \frac{i}{4}\iint I(\theta-\theta_2-\pi)H_0^{(1)}(k|\vec{r}-\vec{\rho}|)I(\theta'-\theta_1)R^2d\theta d\theta'$$
$$= \sum_{\mu,\nu} \frac{i^{\mu+\nu+1}}{4} J_{\mu}(kR)J_{\nu}(kR)H_{\mu+\nu}(kna)\int_0^{2\pi} I(\theta-\theta_2-\pi)e^{-i\mu\theta}Rd\theta \int_0^{2\pi}I(\theta'-\theta_1)e^{i\nu\theta'}Rd\theta'$$
$$=4\sum i^{\mu+\nu+1+(\mu-1)+(-\nu-1)} \frac{J_{\mu}(kR)}{H_{\mu}^{(1)}(kR)} \cdot \frac{J_{\nu}(kR)}{(-1)^{\nu}H_{\nu}^{(1)}(kR)} \cdot H_{\mu+\nu}(nka)e^{-i\mu(\theta_2+\pi)+i\nu\theta_1}$$
$$= -4i\sum_{\mu,\nu} (-1)^{\nu} \frac{J_{\mu}}{H_{\mu}} \frac{J_{\nu}}{H_{\nu}} (kR)H_{\mu+\nu}(nka)e^{-i\mu\theta_2+i\nu\theta_2} \qquad (87)$$

Combining this result with that for the (-n)-th cylinder,

$$g_n = g_{n+} + g_{n-} = -4i \sum_{\mu,\nu} \frac{J_{\mu}}{H_{\mu}} \cdot \frac{J_{\nu}}{H_{\nu}} (kR)H_{\mu+\nu}(kna)\left[(-1)^{\mu}e^{-inskasin\theta_1}+(-1)^{\nu}e^{inskasin\theta_1}\right] \qquad (88)$$

Summing with respect to n,

$$\Delta g = \sum_1^{\infty} g_n = -4i\sum_{\mu,\nu} \frac{J_{\mu}}{H_{\mu}} \frac{J_{\nu}}{H_{\nu}} (kR)\sum_{n=1}^{\infty} H_{\mu+\nu}^{(1)}(nka)\left[(-1)^{\mu}e^{-inskasin\theta_1}+(-1)^{\nu}e^{inskasin\theta_1}\right] \qquad (89)$$

Restricting ourselves in the rest of the discussion to the somewhat simpler case of normal incidence, $\theta_1 = 0$,

$$\Delta g = -4i \sum_{\mu,\nu} \left[(-1)^{\mu}+(-1)^{\nu}\right] \frac{J_\mu}{H_\mu} \cdot \frac{J_\nu}{H_\nu}(kR) \cdot \sum_{n=1}^{\infty} H_{\mu+\nu}(nka) \quad . \tag{89a}$$

3. Evaluation of the Hankel Function Series.

The series involving the ratio of the Bessel to the Hankel function is rapidly convergent for reasonable values of parameters. Hence both the previously considered series for f and the double series are satisfactory for numerical computation. The series $\sum_n H_{\mu+\nu}(nka)$, however, is not only slowly convergent, but also fails to converge absolutely or uniformly with respect to ka. For large n, $H_{\mu+\nu}(nka)$ behaves like $\exp(inka)/\sqrt{n}$. Although the series $\sum \exp(inka)/\sqrt{n}$ can be expressed in terms of the generalized Zeta function of Riemann[4], the latter, unfortunately, is not tabulated. The procedure used here will be to use two series, a linear combination of which can be made to have two terms in the asymptotic expansion identical with those of the Hankel function. Consider

$$y_1(x) = \sum_{n=1}^{\infty} \frac{\Gamma(n+1/2)}{\Gamma(n+1)} x^n \tag{90}$$

and

$$y_2(x) = \sum_{n=1}^{\infty} \frac{\Gamma(n)}{\Gamma(n+1/2)} x^n \tag{91}$$

Using Stirling's theorem, we readily obtain asymptotic series for the Gamma function quotients

$$\frac{\Gamma(n+1/2)}{\Gamma(n+1)} \sim n^{-1/2} \left[1 - \frac{1}{8n} - \frac{13}{384n^2} + \dots \right] \quad , \tag{92a}$$

$$\frac{\Gamma(n)}{\Gamma(n+1/2)} \sim n^{-1/2} \left[1 + \frac{1}{8n} - \frac{13}{384n^2} + \dots \right] \quad , \tag{92b}$$

So that we can approximate $\exp(inka)/\ n^{1/2}$ by

$$\frac{1}{2} \left[\frac{\Gamma(n+1/2)}{\Gamma(n+1)} + \frac{\Gamma(n)}{\Gamma(n+1/2)} \right] \sim n^{-1/2} \left[1 - \frac{13}{384n^2} + \dots \right] \tag{92c}$$

In fact even the first term of the series for $\frac{1}{2}(y_1+y_2)$, n = 1, is

$\frac{1}{2}\left[\Gamma(\frac{3}{2})/\ \Gamma(2) \quad + \quad \Gamma(1)/\ \Gamma(\frac{3}{2})\right] = 1.0073$, only 0.7% off.

Now let us return to the evaluation of the series. Introducing the Beta function, and then using its integral representation[5]

$$y_1(x) = \pi^{-1/2} \sum_{n=1}^{\infty} \frac{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \quad x^n = \pi^{-1/2} \sum_n B(\tfrac{1}{2},\ n+\tfrac{1}{2})\ x^n$$

$$= \pi^{-1/2} \sum \int_0^1 z^{n-1/2}(1-z)^{-1/2} x^n dz$$

$$= \pi^{-1/2} \int_0^1 \frac{dz}{\sqrt{z(1-z)}} \quad \frac{xz}{1-xz} = \sqrt{\pi}\ \left(\frac{1}{\sqrt{1-x}} - 1\right), \qquad (93)$$

and

$$y_2(x) = \pi^{-1/2} \sum \frac{\Gamma(\frac{1}{2})\Gamma(n)}{(n+1/2)}\ x^n = \pi^{-1/2} \sum B(\tfrac{1}{2},n)x^n$$

$$= \pi^{-1/2} \sum \int_0^1 z^{n-1}(1-z)^{-1/2} x^n dz = x\pi^{-1/2} \int_0^1 \frac{dz}{(1-xz)\sqrt{1-z}}$$

$$= \pi^{-1/2} \sqrt{\frac{-x}{1-x}} \quad \log \frac{1+\sqrt{\frac{-x}{1-x}}}{1-\sqrt{\frac{-x}{1-x}}} \qquad (94)$$

Both of these functions are singular at $x = 1$ (or at $ka/2\pi$ an integer) and they behave near the singularity like $(1-x)^{-1/2}$. A parametric plot of $\frac{1}{2}\left[y_1(e^{i\Theta}) + y_2(e^{i\Theta})\right]$ is shown on Fig. 3, for the range $0 < \Theta < \pi$.

Although more complicated procedures will give more accurate results, in this paper $\sum_{n=1}^{\infty} H_{\mu+\nu}(nka)$ will be approximated by*

$(2\pi ka)^{-1/2} e^{i\pi(1/4 + \mu/2 + \nu/2)}\left[y_1(e^{ika}) + y_2(e^{ika})\right]$. Eq. (89a) becomes, then,

$$\Delta g \approx -2\left(\frac{2i}{\pi ka}\right)^{1/2} \left\{ \sum_{\mu,\nu} \left[(-1)^{\mu}+(-1)^{\nu}\right] e^{\frac{-i(\mu+\nu)\pi}{2} - i\mu\theta_2}\ \frac{J_\mu}{H_\mu} \cdot \frac{J_\nu}{H_\nu}(kR) \right\} \left[y_1(e^{ika})+y_2(e^{ika})\right] , \quad (9$$

Finally, the right hand member of Eq. (19) becomes

$$\frac{-i}{2a} \frac{-4i \sum_m \left[J_m(kR)/H_m(kR)\right] e^{im\Theta_2}}{1 + \frac{\sum_{\mu,\nu}\left[(-1) +(-1)^{\nu}\right] e^{-i(\mu+\nu)\pi/2 - in\Theta_2} (J_\mu/H_\mu)(J_\nu/H_\nu)(kR)\left[y_1(e^{ika})+y_2(e^{ika})\right]}{\sqrt{2\pi ika}\ \left[\sum_n J_n(kR)/H_n(kR)\ e^{in\Theta_2}\right]}} \qquad (96$$

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* The approximating function and the Hankel function series have the same first term in the asymptotic expansion.

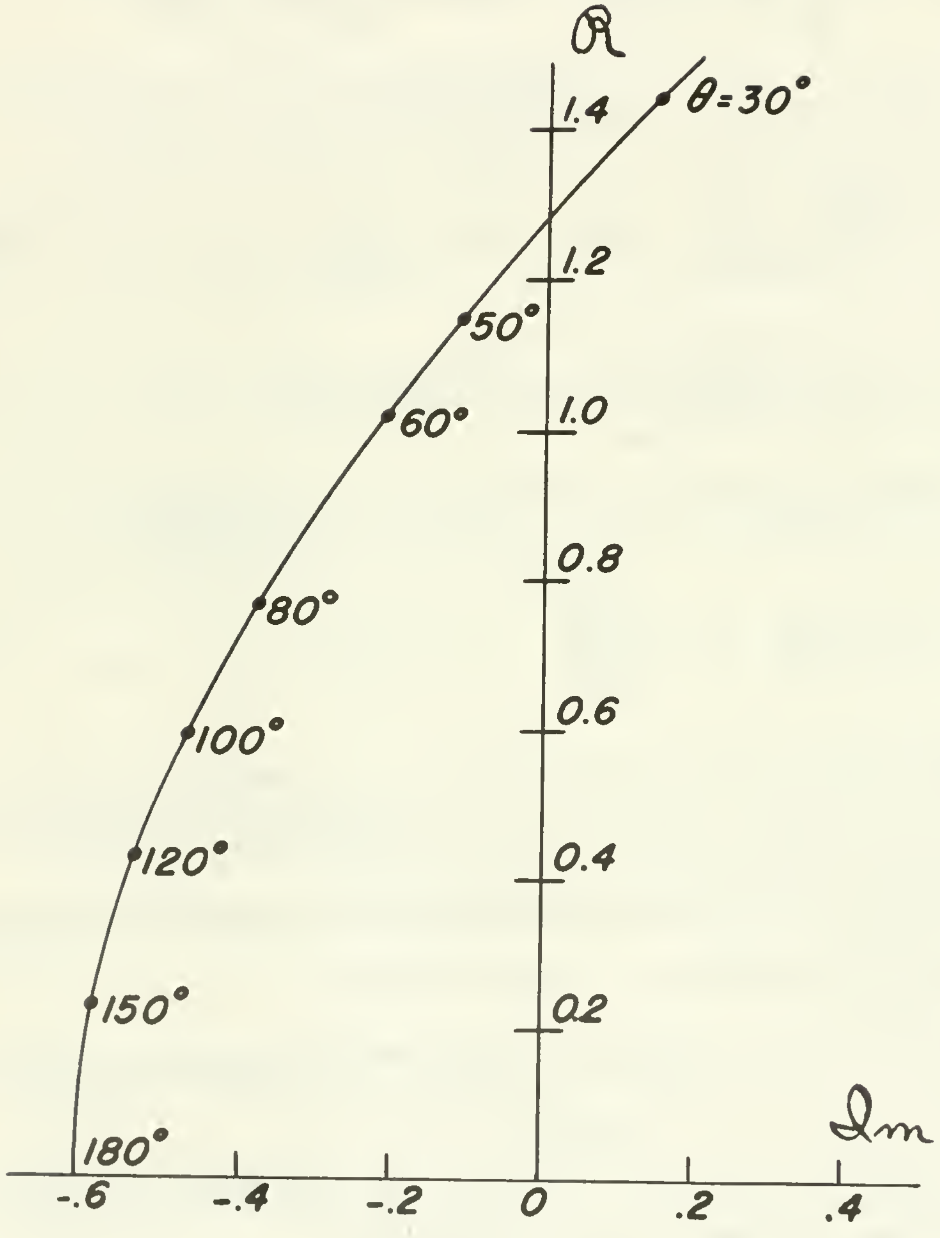

Fig. 3. A parametric plot of $\frac{1}{2}\left[y_1(e^{i\theta}) + y_2(e^{i\theta})\right]$

At this point we may note that the quantity of particular interest in (96) is the correction factor

$$C_{TE} \approx 1 + \frac{\sum\limits_{\mu,\nu} \left[(-1)^{\mu} + (-1)^{\nu}\right] e^{-i(\mu+\nu)\pi/2 - in\theta_2} \frac{J_\mu}{H_\mu} \cdot \frac{J_\nu}{H_\nu}(kR)\left[y_1(e^{ika}) + y_2(e^{ika})\right]}{\sqrt{2\pi ika}\left[\sum\limits_n \frac{J_n}{H_n}(kR)e^{in\theta_2}\right]} \tag{97}$$

since the transmission coefficients based on no interaction theory have been treated before. Near critical points $k_c a = 2\pi n$, the correction factor dominates the behavior of the transmission coefficients, which go to zero like $(k-k_c)^{1/2}$.

4. TM Wave Problem

Let us take up now the TM case. We derive the trial current distribution as above, except that the application of the boundary condition that the normal derivative vanish gives us, instead of (78),

$$i^n J_n'(kR) + A_n\left[H_n^{(1)}(kR)\right]' = 0 \quad . \tag{98}$$

The current distribution is now proportional to the function on the boundary

$$I(\theta) = \sum_{n=-\infty}^{\infty} \frac{2i^{n-1}}{\pi kRH_n'(kR)} e^{in\theta} \quad . \tag{99}$$

The normalizing integrals turn out to be

$$f = -4i \sum_{-\infty}^{\infty} \frac{J_n'(kR)}{H_n'(kR)} e^{in(\theta_1-\theta_2)} \tag{100}$$

instead of (83).

Proceeding to the evaluation of the Green's function integral, we note that

$$(G^0)_{\nu\nu'} = \frac{i}{4} k^2 \sum_m H_m'(kR) J_m'(kR) e^{im(\theta-\theta')} \tag{101}$$

so that, carrying out the integration as in (85), we obtain

$$g_0 = -4i \sum \frac{J_m'(kR)}{H_m'(kR)} e^{im(\theta_1-\theta_2)} \quad , \tag{102}$$

again identical with f, for the same reason.

For the remaining terms we have

$$(G^n)_{\nu\nu'} = \frac{1}{4} k^2 \sum_{\mu,\nu} i^{(\mu+\nu)} J'_\mu(kR) J'_\nu(kR) H_{\mu+\nu}(kna) e^{-i\mu\theta + i\nu\theta' + inskasin\theta_1} \tag{103}$$

so that,

$$g_n = -4i \sum_{\mu,\nu} \left[(-1)^\mu e^{-inskasin\theta_1} + (-1)^\nu e^{inskasin\theta_1}\right] \frac{J'_\mu}{H'_\mu} \cdot \frac{J'_\nu}{H'_\nu}(kR) H_{\mu+\nu}(nka) \tag{104}$$

In the case of normal incidence, the correction factor is then

$$C_{TM} \approx 1 + \frac{\sum_{\mu,\nu} \left[(-1)^\mu + (-1)^\nu\right] e^{-i(\mu+\nu)\pi/2 - i\mu\theta_2} \frac{J'_\mu}{H'_\mu} \cdot \frac{J'_\nu}{H'_\nu}(kR) \left[y_1(e^{ika}) + y_2(e^{ika})\right]}{\sqrt{2\pi ika} \left[\sum_n \frac{J'_n}{H'_n}(kR) e^{in\theta_2}\right]} \tag{105}$$

5. Discussion of Results

A sample calculation was performed for kR = 2, ka = 8; this is the case of the half open grating so often used in infrared spectroscopy, with only one order of diffraction at an angle 51.76°. The correction factor for the TE case was found to be 1.05, thus indicating that for most practical cases the no interaction assumption is perhaps satisfactory. It is unfortunate that the numerical results based on this assumption are not available. Most authors having arrived at the series (83), state that it does not converge rapidly, and make further approximations. As a result the experimenter is left with the very crude Kirchhoff approximation which predicts zero intensity of even order spectra in the half open grating. For the case of cylindrical grating wires with kR = 4, ka = 16, the intensity of the second order wave (also at an angle 51.76°) was estimated and found not to be negligible compared to first order spectrum intensity for the case kR = 2, ka = 8. Thus, in millimeter wave spectroscopy, the contamination due to second order spectra cannot be neglected.

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